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Coconvex Polynomial Approximation of Twice Differentiable Functions

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For a function $f \in C^2[-1,1]$ with $1 \leqslant r < \infty$ inflection points and sufficiently large n we construct an algebraic polynomial p_n of degree $\leqslant n$ satisfying $f''(x)\,p_n''(x) \geqslant 0$, $x \in [-1,1]$, and such that $\|f^{(\nu)} - p_n^{(\nu)}\|_\infty \leqslant C_\nu n^{-2+\nu} \omega_\varphi(f'', n^{-1})$, $\nu = 0, 1, 2$, where $C_\nu = C_\nu(r)$, $\nu = 0, 1$, $C_2 = C_2(r)/\sqrt{1-\alpha^2}$ (α is the point of inflection nearest to ± 1), and $\omega_\varphi(f'', n^{-1})$ denotes the Ditzian–Totik modulus of continuity of f'' in the uniform metric.

1. INTRODUCTION AND MAIN RESULT

Our prime interest in this paper is comonotone and coconvex polynomial approximation, that is, approximation of a function f which is piecewise monotone (or piecewise convex, i.e., has finitely many inflection points) by polynomials which are comonotone (or coconvex) with f. Let Π_n denote the set of all algebraic polynomials of degree $\leqslant n$, $\|\cdot\| := \|\cdot\|_\infty$ denote the uniform norm and $\varphi(x) := \sqrt{1-x^2}$. Recall that the m-th order Ditzian–Totik modulus $\omega_\varphi^m(f, \delta)$ and the usual modulus of smoothness $\omega^m(f, \delta)$ are given respectively by (see [2], for example)

$$\omega_\varphi^m(f, \delta) = \sup_{0 < h \leqslant \delta} \|\Delta_{h\varphi(x)}^m(f, x)\| \qquad \text{and} \qquad \omega^m(f, \delta) = \sup_{0 < h \leqslant \delta} \|\Delta_h^m(f, x)\|,$$

where

$$\Delta_\eta^m(f, x) := \begin{cases} \sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} f\left(x - \frac{m}{2}\eta + i\eta\right), & \text{if } x \pm \frac{m}{2}\eta \in [-1, 1], \\ 0, & \text{otherwise.} \end{cases}$$

is the symmetric m-th difference.

The following result on comonotone approximation of continuous functions is known.

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THEOREM A. *Let $f \in C[-1,1]$ have $1 \leqslant r < \infty$ changes of monotonicity at the points $\{y_i\}_{i=1}^r$: $-1 < y_1 < \cdots < y_r < 1$. Then there exist polynomials p_n^*, $p_n^{**} \in \Pi_n$ which are comonotone with f on $[-1,1]$ and such that*

$$\|f - p_n^*\| \leqslant C^*(r, d(r))\, \omega^2(f, n^{-1}) \tag{1}$$

and

$$\|f - p_n^{**}\| \leqslant C^{**}(r, d_0)\, \omega_\varphi(f, n^{-1}), \tag{2}$$

where $d(r) := \min\{y_1 + 1, y_2 - y_1, ..., y_r - y_{r-1}, 1 - y_r\}$ and $d_0 := \min\{y_1 + 1, 1 - y_r\}$.

For piecewise monotone differentiable functions we have the following.

THEOREM B. *Let $f \in C^1[-1,1]$ have $1 \leqslant r < \infty$ changes of monotonicity at the points $\{y_i\}_{i=1}^r$: $-1 < y_1 < \cdots < y_r < 1$. Then for each $n \geqslant 1$ there is a polynomial $p_n \in \Pi_n$ comonotone with f and such that*

$$\|f - p_n\| \leqslant C(r, d_0)\, n^{-1} \omega_\varphi(f', n^{-1}) \tag{3}$$

and

$$\|f' - p_n'\| \leqslant C(r, d_0)\, \omega_\varphi(f', n^{-1}), \tag{4}$$

where $d_0 := \min\{y_1 + 1, 1 - y_r\}$.

Theorem B and the estimate (2) in Theorem A were proved by D. Leviatan [5]. Estimate (1) is due to A. S. Shvedov [13] and X. M. Yu [16]. It was also shown by A. S. Shvedov [13] that the constant C^* in (1) can not be replaced by that independent of $d(r)$. Moreover, the estimate (1) is exact in the sense that ω^2 can not be replaced by ω^3. This is an immediate consequence of S. P. Zhou [17].

Other relevant results can be found in [1, 3, 6–12], for example.

Thus, for comonotone polynomial approximation there are quite a few satisfactory results. At the same time, it seems that little is known about coconvex approximation. The only direct results of this type which we are aware of at present are the following.

(i) R. K. Beatson and D. Leviatan remarked in [1] that it is possible to obtain Jackson type theorems for coconvex approximation of functions with only one inflection point.

(ii) X. M. Yu [15] obtained a Jackson type estimate of coconvex approximation of a function with one *regular convexity-turning* point.

(iii) Also, in [15] X. M. Yu quoted her result on coconvex approximation of differentiable functions (which are at least in $C^3[-1, 1]$) with some extra conditions on convexity-turning points.

The goal of this paper is to present a result on coconvex approximation which is analogous to Theorem B. Namely, we prove the following theorem.

THEOREM 1 (Coconvex Approximation). *Let* $f \in C^2[-1, 1]$ *have* $1 \leqslant r < \infty$ *inflection points at* $\{y_i\}_{i=1}^r$: $-1 < y_1 < \cdots < y_r < 1$, $d_0 := \min\{y_1 + 1, 1 - y_r\}$ *and* $d(r) := \min\{y_1 + 1, y_2 - y_1, ..., y_r - y_{r-1}, 1 - y_r\}$. *Then there exists a constant* $A = A(r)$ *such that for each* $n > A(r)/d(r)$ *there is a polynomial* $p_n \in \Pi_n$ *satisfying* $f''(x)\, p_n''(x) \geqslant 0$, $x \in [-1, 1]$ *and such that*

$$\|f - p_n\| \leqslant C(r)\, n^{-2} \omega_\varphi(f'', n^{-1}), \tag{5}$$

$$\|f' - p_n'\| \leqslant C(r)\, n^{-1} \omega_\varphi(f'', n^{-1}) \tag{6}$$

and

$$\|f'' - p_n''\| \leqslant \frac{C(r)}{\sqrt{d_0}}\, \omega_\varphi(f'', n^{-1}). \tag{7}$$

COROLLARY 2 (Comonotone Approximation). *Let f be the same as in Theorem B. Then there exists a constant* $A = A(r)$ *such that for each* $n > A(r)/d(r)$ *there is a polynomial* $p_n \in \Pi_n$ *satisfying* $f'(x)\, p_n'(x) \geqslant 0$, $x \in [-1, 1]$, *and the following inequalities hold*:

$$\|f - p_n\| \leqslant C(r)\, n^{-1} \omega_\varphi(f', n^{-1}) \tag{8}$$

and

$$\|f' - p_n'\| \leqslant \frac{C(r)}{\sqrt{d_0}}\, \omega_\varphi(f', n^{-1}). \tag{9}$$

2. DEFINITIONS AND NOTATION

Throughout this paper we use the following notation (cf. [4, 11, 12]):

$$I := [-1, 1]; \qquad \Delta_n(x) := \frac{\sqrt{1 - x^2}}{n} + \frac{1}{n^2};$$

$$x_j := x_{j,n} := \cos\frac{j\pi}{n}, \quad 0 \leqslant j \leqslant n; \qquad \bar{x}_j := \bar{x}_{j,n} := \cos\left(\frac{j\pi}{n} - \frac{\pi}{2n}\right), \quad 1 \leqslant j \leqslant n;$$

$$x_j^0 := x_{j,n}^0 := \cos\left(\frac{j\pi}{n} - \frac{\pi}{4n}\right) \quad \text{if } j < n/2,$$

$$x_j^0 := x_{j,n}^0 := \cos\left(\frac{j\pi}{n} - \frac{3\pi}{4n}\right) \quad \text{if } j \geqslant n/2;$$

$$I_j := I_{j,n} := [x_j, x_{j-1}], \qquad h_j := h_{j,n} := x_{j-1} - x_j, \qquad 1 \leqslant j \leqslant n$$

(note that $h_{j\pm 1} < 3h_j$ and $\Delta_n(x) < h_j < 5\Delta_n(x)$ for $x \in I_j$).

Also,

$$t_j(x) := t_{j,n}(x) := (x - x_j^0)^{-2} \cos^2 2n \arccos x + (x - \bar{x}_j)^{-2} \sin^2 2n \arccos x$$

is the algebraic polynomial of degree $4n - 2$ (see [11, 12], for example).

If

$$\Pi_j(n, \mu; a_1, ..., a_m; b_1, ..., b_k) := \int_{-1}^{1} t_j(y)^\mu \prod_{i=1}^{m} (y - a_i) \prod_{i=1}^{k} (b_i - y)\, dy,$$

then for $a_i \leqslant x_j$, $1 \leqslant i \leqslant m$, $b_i \geqslant x_{j-1}$, $1 \leqslant i \leqslant k$ and sufficiently large μ

$$T_j(n, \mu; a_1, ..., a_m; b_1, ..., b_k)(x) := \frac{\int_{-1}^{x} t_j(y)^\mu \prod_{i=1}^{m} (y - a_i) \prod_{i=1}^{k} (b_i - y)\, dy}{\Pi_j(n, \mu; a_1, ..., a_m; b_1, ..., b_k)}$$

is the algebraic polynomial of degree $2\mu(2n-1) + m + k + 1$, which is well defined because $\Pi_j(n, \mu; a_1, ..., a_m; b_1, ..., b_k)$ is positive for large μ (see Proposition 4).

If $m = 0$ or $k = 0$, i.e., if there are no a_i's or b_i's in the definition of T_j, we use the notation $T_j(n, \mu; \varnothing; b_1, ..., b_k)(x)$ or $T_j(n, \mu; a_1, ..., a_m; \varnothing)(x)$, respectively. Thus, for example,

$$T_j(n, \mu; \varnothing; b_1, ..., b_k)(x) := \frac{\int_{-1}^{x} t_j(y)^\mu \prod_{i=1}^{k} (b_i - y)\, dy}{\int_{-1}^{1} t_j(y)^\mu \prod_{i=1}^{k} (b_i - y)\, dy}.$$

We also denote

$$\psi_j := \psi_{j,n} := \psi_{j,n}(x) := \frac{h_j}{|x - x_j| + h_j}, \qquad \chi_j(x) := \begin{cases} 1 & \text{if } x \geqslant x_j, \\ 0 & \text{otherwise}, \end{cases}$$

$$\operatorname{sgn}(f(x)) := \begin{cases} -1 & \text{if } f(x) < 0, \\ 0 & \text{if } f(x) = 0, \\ 1 & \text{if } f(x) > 0. \end{cases} \qquad \text{and} \qquad \operatorname{sgn}_\alpha(x) := \operatorname{sgn}(x - \alpha).$$

C are positive constants which are not necessarily the same even when they occur on the same line. In order to emphasize that C depends only on the parameters $\nu_1, ..., \nu_k$ we use the notation $C(\nu_1, ..., \nu_k)$. At the same time, $A(r)$ denote constants which depend only on r and remain fixed throughout the paper.

3. Auxiliary Statements and Results

Proposition C (See [4, 11, 12], for Example). *The following inequalities hold*:

$$\min\{(x-x_j^0)^{-2}, (x-\bar{x}_j)^{-2}\} \leqslant t_j(x) \leqslant \max\{(x-x_j^0)^{-2}, (x-\bar{x}_j)^{-2}\}, \quad x \in I, \tag{10}$$

$$t_j(x) \leqslant 10^3 h_j^{-2}, \qquad x \in I_j, \tag{11}$$

$$x_j^0 - x_j > \frac{\bar{x}_j - x_j}{2} > \frac{1}{4} h_j, \quad x_{j-1} - \bar{x}_j > \frac{1}{4} h_j, \quad \bar{x}_j - x_j^0 \leqslant \frac{3}{8} h_j \quad \text{for } j \leqslant \frac{n}{2}, \tag{12}$$

$$x_{j-1} - x_j^0 > \frac{x_{j-1} - \bar{x}_j}{2} > \frac{1}{4} h_j, \quad \bar{x}_j - x_j > \frac{1}{4} h_j, \quad x_j^0 - \bar{x}_j \leqslant \frac{3}{8} h_j \quad \text{for } j > \frac{n}{2} \tag{13}$$

and

$$(|x-x_j| + h_j)^{-2} \leqslant t_j(x) \leqslant 4 \cdot 10^3 (|x-x_j| + h_j)^{-2}, \qquad x \in I. \tag{14}$$

The following proposition can be easily verified either by straightforward computations or by induction on ν.

Proposition 3. *Let ν be an integer, $p \geqslant \nu + 2$ and $c_i \geqslant t_0 > 0$, $1 \leqslant i \leqslant \nu$. Then the following estimates are valid*:

$$\frac{1}{p-1} c_1 \cdots c_\nu t_0^{1-p} \leqslant \int_{t_0}^{\infty} t^{-p} \prod_{i=1}^{\nu} (c_i + t)\, dt \leqslant \frac{2^\nu}{p-\nu-1} c_1 \cdots c_\nu t_0^{1-p}.$$

The following result is a generalization of Proposition 2 of [4].

Proposition 4. *Let $1 \leqslant j \leqslant n$ be fixed. Then the following inequalities hold*:

$$C(\mu) \leqslant \Pi_j(n, \mu; a_1, ..., a_m; b_1, ..., b_k)\, h_j^{2\mu-1} \left(\prod_{i=1}^{m} (x_{j-1} - a_i) \prod_{i=1}^{k} (b_i - x_j) \right)^{-1}$$
$$\leqslant C(\mu),$$

where $a_i \leqslant x_j$, $1 \leqslant i \leqslant m$, $b_i \geqslant x_{j-1}$, $1 \leqslant i \leqslant k$ *and* μ *is sufficiently large in comparison with* k *and* m *(for example,* $\mu \geqslant 5(m+k+1)$ *will do).*

Proof (cf. [4, 11]). The proposition will be proved for $j \leqslant n/2$. For $j > n/2$ the proof is analogous with the only difference that instead of (12) one should use (13).

We write

$$\Pi_j(n, \mu; a_1, \ldots, a_m; b_1, \ldots, b_k) = \left\{\int_{-1}^{x_j} + \int_{x_j}^{x_{j-1}} + \int_{x_{j-1}}^{1}\right\} = \Theta_1 + \Theta_2 + \Theta_3.$$

Now denoting $\prod_{i=1}^{m} (x_{j-1} - a_i) \prod_{i=1}^{k} (b_i - x_j)$ by Γ_{jmk} and using estimates (10)–(12), we have

$$\Theta_2 \leqslant \Gamma_{jmk}(x_{j-1} - x_j)\, 10^{3\mu} h_j^{-2\mu} = 10^{3\mu} \Gamma_{jmk} h_j^{-2\mu+1}$$

and

$$\begin{aligned}
\Theta_2 &\geqslant \int_{x_j^0}^{\bar{x}_j} \prod_{i=1}^{m} (x_j^0 - a_i) \prod_{i=1}^{k} (b_i - \bar{x}_j) \min\{(y - x_j^0)^{-2\mu}, (y - \bar{x}_j)^{-2\mu}\}\, dy \\
&\geqslant 4^{-m-k} \Gamma_{jmk} \int_{x_j^0}^{\bar{x}_j} \min\{(y - x_j^0)^{-2\mu}, (y - \bar{x}_j)^{-2\mu}\}\, dy \\
&\geqslant \frac{2}{2\mu - 1} 4^{-m-k} (2^{2\mu-1} - 1)\, \Gamma_{jmk} (\bar{x}_j - x_j^0)^{-2\mu+1} \\
&\geqslant \frac{2}{2\mu - 1} 4^{-m-k} (2^{2\mu-1} - 1) \left(\frac{8}{3}\right)^{2\mu-1} \Gamma_{jmk} h_j^{-2\mu+1}.
\end{aligned}$$

Similarly, Proposition 3 and the inequalities (12) yield

$$\begin{aligned}
|\Theta_1| &\leqslant \int_{-1}^{x_j} t_j(y)^{\mu} \prod_{i=1}^{m} |y - a_i| \prod_{i=1}^{k} (b_i - y)\, dy \\
&\leqslant \int_{-1}^{x_j} (x_j^0 - y)^{-2\mu} \prod_{i=1}^{m} (|x_{j-1} - a_i| + x_j^0 - y) \prod_{i=1}^{k} (|b_i - x_j| + x_j^0 - y)\, dy \\
&\leqslant \int_{x_j^0 - x_j}^{\infty} t^{-2\mu} \prod_{i=1}^{m} (|x_{j-1} - a_i| + t) \prod_{i=1}^{k} (|b_i - x_j| + t)\, dt \\
&\leqslant \frac{2^{m+k}}{2\mu - m - k - 1} \Gamma_{jmk} (x_j^0 - x_j)^{-2\mu+1} \\
&\leqslant 4^{2\mu-1} \frac{2^{m+k}}{2\mu - m - k - 1} \Gamma_{jmk} h_j^{-2\mu+1}
\end{aligned}$$

and

$$
\begin{aligned}
|\Theta_3| &\leqslant \int_{x_{j-1}}^{1} t_j(y)^{\mu} \prod_{i=1}^{m} (y-a_i) \prod_{i=1}^{k} |b_i - y|\, dy \\
&\leqslant \int_{x_{j-1}}^{1} (y-\bar{x}_j)^{-2\mu} \prod_{i=1}^{m} (|x_{j-1}-a_i| + y - \bar{x}_j) \prod_{i=1}^{k} (|b_i - x_j| + y - \bar{x}_j)\, dy \\
&\leqslant \int_{x_{j-1}-\bar{x}_j}^{\infty} t^{-2\mu} \prod_{i=1}^{m} (|x_{j-1}-a_i| + t) \prod_{i=1}^{k} (|b_i - x_j| + t)\, dt \\
&\leqslant \frac{2^{m+k}}{2\mu - m - k - 1} \Gamma_{jmk} (x_{j-1} - \bar{x}_j)^{-2\mu+1} \\
&\leqslant 4^{2\mu-1} \frac{2^{m+k}}{2\mu - m - k - 1} \Gamma_{jmk} h_j^{-2\mu+1}.
\end{aligned}
$$

Hence,

$$
\Pi_j(n, \mu; a_1, ..., a_m; b_1, b_k)\, h_j^{2\mu-1} (\Gamma_{jmk})^{-1} \leqslant 10^{3\mu} + \frac{2^{4\mu+m+k-1}}{2\mu - m - k - 1} \leqslant 10^{3\mu+1}.
$$

Finally, the inequalities in the other direction are the following

$$
\begin{aligned}
&\Pi_j(n, \mu; a_1, ..., a_m; b_1, ..., b_k)\, h_j^{2\mu-1} (\Gamma_{jmk})^{-1} \\
&\quad \geqslant \frac{2}{2\mu-1} 4^{-m-k} (2^{2\mu-1} - 1) \left(\frac{8}{3}\right)^{2\mu-1} - \frac{2^{4\mu+m+k-1}}{2\mu - m - k - 1} \geqslant \frac{1}{\mu}. \quad \blacksquare
\end{aligned}
$$

LEMMA 5. *Let $a_i \leqslant x_j$, $1 \leqslant i \leqslant m$, $b_i \geqslant x_{j-1}$, $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant n$ be a fixed index. Then for the polynomial $T_j(x) := T_j(n, \mu; a_1, ..., a_m; b_1, ..., b_k)(x)$ of degree $\leqslant 4\mu n + m + k$ the following inequalities hold:*

$$
|T_j'(x)| \leqslant C(\mu)\, \psi_j^{2\mu-m-k} h_j^{-1} \tag{15}
$$

and

$$
|\chi_j(x) - T_j(x)| \leqslant C(\mu)\, \psi_j^{2\mu-m-k-1}, \tag{16}
$$

where $\mu \geqslant 5(m+k+1)$, $x \in I$.

Proof. Proposition 4 and the inequalities (14) imply for any $x \in I$

$$|T_j'(x)| \leqslant C(\mu)\, t_j(x)^\mu\, h_j^{2\mu-1} \prod_{i=1}^{m} \frac{|x-a_i|}{|x_{j-1}-a_i|} \prod_{i=1}^{k} \frac{|b_i-x|}{|b_i-x_j|}$$

$$\leqslant C(\mu)\, \psi_j^{2\mu} h_j^{-1} \prod_{i=1}^{m} \left(1+\frac{|x-x_j|+h_j}{h_j}\frac{h_j}{|x_{j-1}-a_i|}\right)$$

$$\times \prod_{i=1}^{k} \left(1+\frac{|x-x_j|+h_j}{h_j}\frac{h_j}{|b_i-x_j|}\right)$$

$$\leqslant C(\mu)\, \psi_j^{2\mu} h_j^{-1}(1+\psi_j^{-1})^{m+k} \leqslant C(\mu)\, \psi_j^{2\mu-m-k} h_j^{-1},$$

which is the inequality (15).

To prove the remaining inequality (16), first, we consider the case $x < x_j$. The estimate (15) implies

$$|\chi_j(x)-T_j(x)| = |T_j(x)| = \left|\int_{-1}^{x} T_j'(y)\, dy\right|$$

$$\leqslant C(\mu)\, h_j^{2\mu-m-k-1} \int_{-\infty}^{x} (x_j-y+h_j)^{-2\mu+m+k}\, dy$$

$$\leqslant C(\mu)\, \psi_j^{2\mu-m-k-1}.$$

For $x \geqslant x_j$, similarly, we have

$$|\chi_j(x)-T_j(x)| = |1-T_j(x)| = \left|\int_{x}^{1} T_j'(y)\, dy\right|$$

$$\leqslant C(\mu)\, h_j^{2\mu-m-k-1} \int_{x}^{\infty} (y-x_j+h_j)^{-2\mu+m+k}\, dy$$

$$\leqslant C(\mu)\, \psi_j^{2\mu-m-k-1}.$$

Thus, the inequality (16) is also verified. ∎

Using the identity $\operatorname{sgn}_{x_j}(x) = 2\chi_j(x)-1$ a.e. we conclude that the polynomial $\tilde{T}_j(x) := 2T_j(x)-1$ sufficiently approximates $\operatorname{sgn}_{x_j}(x)$. Also, it is easy to see that $\tilde{T}_j(x)$ is increasing on I_j. Later on we will need similar polynomial (it will be denoted by $Q_j(x)$) which satisfies one extra condition: *sgn*$(Q_j(x)) =$ *sgn*$_\alpha(x)$ *for some* $\alpha \in I_j$ (In other words, we want the polynomial not only approximate $\operatorname{sgn}_\alpha(x)$ and be increasing on I_j, but also be copositive with $\operatorname{sgn}_\alpha(x)$). Our construction of $\tilde{T}_j(x)$ does not immediately yield this equality. However, $\tilde{T}_j(x)$ can be refined to satisfy it. Namely, the following lemma is valid (note that we assume $a_i \leqslant x_{j+1}$, $1 \leqslant i \leqslant m$, $b_i \geqslant x_{j-2}$, $1 \leqslant i \leqslant k$).

LEMMA 6. *Let n and $1 \leqslant j \leqslant n$ be fixed, $a_i \leqslant x_{j+1}$, $1 \leqslant i \leqslant m$, $b_i \geqslant x_{j-2}$, $1 \leqslant i \leqslant k$ and $\alpha \in I_j$. Then there exist numbers $A = 2^\nu$, $\nu \in \mathcal{N}$ and $0 \leqslant \xi \leqslant 1$ such that for the polynomial*

$$\begin{aligned} Q_j(x) &:= Q_j(n, \mu, \alpha; a_1, ..., a_m; b_1, ..., b_k)(x) \\ &= 2\{\xi T_{j_1}(An, \mu; a_1, ..., a_m; b_1, ..., b_k)(x) \\ &\quad + (1-\xi) T_{j_2}(An, \mu; a_1, ..., a_m; b_1, ..., b_k)(x)\} - 1, \end{aligned}$$

where indices $j_1 = j_1(A)$ and $j_2 = j_2(A)$ are chosen so that $x_{j_1, An} = \bar{x}_{j+1}$ and $x_{j_2, An} = \bar{x}_{j-1}$, the following is true

$$Q_j'(x) \prod_{i=1}^{m} (x - a_i) \prod_{i=1}^{k} (b_i - x) \geqslant 0,$$

$$\text{in particular, } Q_j(x) \text{ increases on } I_{j+1} \cup I_j \cup I_{j-1}, \tag{17}$$

$$|sgn_\alpha(x) - Q_j(x)| \leqslant C(\mu)\, \psi_j^{2\mu - m - k - 1}, \tag{18}$$

$$|sgn_\alpha(x) - Q_j(x)| \leqslant 2/3,\ x \notin I_{j+1} \cup I_j \cup I_{j-1}, \tag{19}$$

$$Q_j(x)\, sgn_\alpha(x) \geqslant 0 \qquad \text{for all} \quad x \in [-1, 1] \tag{20}$$

and

$$|Q_j'(x)| \leqslant C(\mu)\, \psi_j^{2\mu - m - k} h_j^{-1}. \tag{21}$$

Proof. First of all, denoting $n_1 := An$ we obtain the following consequence of Lemma 5 for any $x \notin I_{j+1}$ (note that $\chi_{j_1, n_1}(x) = \chi_j(x)$ for $x \notin I_{j+1}$):

$$\begin{aligned} &|\chi_j(x) - T_{j_1}(x)| \\ &\leqslant C(\mu) \left(\frac{h_{j_1, n_1}}{|x - \bar{x}_{j+1}| + h_{j_1, n_1}} \right)^{2\mu - m - k - 1} \leqslant C(\mu) \left(\frac{h_{j_1, n_1}}{h_j/12 + h_{j_1, n_1}} \right)^{2\mu - m - k - 1} \\ &\leqslant C(\mu) \left(\frac{h_{j_1, n_1}}{h_j} \right)^{2\mu - m - k - 1} \leqslant C(\mu) \left(\frac{n}{n_1} \right)^{2\mu - m - k - 1} = \frac{C(\mu)}{A^{2\mu - m - k - 1}} \leqslant \frac{1}{3} \end{aligned}$$

for sufficiently large A.

Analogously, for any $x \notin I_{j-1}$ choosing n_1 to be large in comparison with n one has

$$\begin{aligned} |\chi_{j-1}(x) - T_{j_2}(x)| &\leqslant C(\mu) \left(\frac{h_{j_2, n_1}}{|x - \bar{x}_{j-1}| + h_{j_2, n_1}} \right)^{2\mu - m - k - 1} \\ &\leqslant C(\mu) \left(\frac{h_{j_2, n_1}}{h_j} \right)^{2\mu - m - k - 1} \leqslant C(\mu) \left(\frac{n}{n_1} \right)^{2\mu - m - k - 1} \\ &= \frac{C(\mu)}{A^{2\mu - m - k - 1}} \leqslant \frac{1}{3}. \end{aligned}$$

Now let A be fixed and such that the above inequalities are satisfied.

Since $\alpha \in I_j$ we have, in particular, $T_{j_1}(\alpha) \geqslant 2/3$ and $T_{j_2}(\alpha) \leqslant 1/3$. Hence, there exists $0 \leqslant \xi \leqslant 1$ such that $Q_j(\alpha) = 2\{\xi T_{j_1}(\alpha) + (1-\xi) T_{j_2}(\alpha)\} - 1 = 0$.

The above estimates yield

$$|\mathrm{sgn}_\alpha(x) - Q_j(x)| \leqslant 2/3, \; x \notin I_{j+1} \cup I_j \cup I_{j-1},$$

which is the inequality (19).

Now note that (18) and (21) are immediate corollaries of Lemma 5, the fact that $\psi_j \sim C$, $x \in I_{j+1} \cup I_j \cup I_{j-1}$ and the observation that $\max\{\psi_{j_1,n_1}, \psi_{j_2,n_1}\} \leqslant 10A^2\psi_j$ (see [4, ineq. (62)], for example).

Finally, using the definitions of $Q_j(x)$ and $T_j(x)$ we have

$$\begin{aligned}
\frac{Q_j'(x)}{2} &= \{\xi T_{j_1}'(An, \mu; a_1, ..., a_m; b_1, ..., b_k)(x) \\
&\quad + (1-\xi) T_{j_2}'(An, \mu; a_1, ..., a_m; b_1, ..., b_k)(x)\} \\
&= \prod_{i=1}^{m} (x - a_i) \prod_{i=1}^{k} (b_i - x) \\
&\quad \times \left[\frac{\xi t_{j_1, An}(x)^\mu}{\Pi_{j_1}(An, \mu; a_1, ..., a_m; b_1, ..., b_k)} + \frac{(1-\xi)\, t_{j_2, An}(x)^\mu}{\Pi_{j_2}(An, \mu; a_1, ..., a_m; b_1, ..., b_k)}\right].
\end{aligned}$$

Since the expression in square brackets is always positive (see inequalities (14) and Proposition 4) we conclude that $Q_j'(x)$ is copositive with $\prod_{i=1}^m (x-a_i) \prod_{i=1}^k (b_i - x)$. In particular, since $a_i \leqslant x_{j+1}$, $1 \leqslant i \leqslant m$ and $b_i \geqslant x_{j-2}$, $1 \leqslant i \leqslant k$, $Q_j(x)$ increases on $I_{j+1} \cup I_j \cup I_{j-1}$. Together with (19) this implies (20). ▌

4. Proof of Theorem 1

We use the method from [1] and [5] and prove Theorem 1 by induction on r, the number of inflection points. For $r=0$ Theorem 1 becomes a theorem on convex approximation which is a simple consequence of Theorem 2 of [4]. Let us assume that (5)–(7) are valid for functions with $r-1 \geqslant 0$ inflection points. Let $f \in C^2[-1, 1]$ have $r < \infty$ inflection points at $\{y_i\}_{i=1}^r$: $-1 < y_1 < \cdots < y_r < 1$. Without loss of generality we can assume that $f''(x) \geqslant 0$, $x \in [-1, y_1]$. We fix one of y_i's. In fact, it is not important which one to fix, but notation and considerations are simpler for $y_1 =: \alpha$ (also, if $r = 1$ it is convenient to denote $y_2 := y_r := 1$). We can assume that $f(\alpha) = f'(\alpha) = 0$ (subtract a linear function from f which, obviously, has no effect on convexity). Since $f \in C^2$ and α is an inflection point, then $f''(\alpha) = 0$.

Following [1] we define the "flipped" function

$$\hat{f}(x) := \begin{cases} f(x) & \text{if } x \geqslant \alpha, \\ -f(x) & \text{if } x < \alpha. \end{cases}$$

Then $\hat{f} \in C^2[-1, 1]$, $\hat{f}(\alpha) = \hat{f}'(\alpha) = \hat{f}''(\alpha) = 0$ and $\hat{f}$ has $r-1$ inflection points at $y_2, ..., y_r$, and also, as was shown in [5],

$$\omega_\varphi(\hat{f}'', t) \leqslant C\omega_\varphi(f'', t), \; t > 0. \tag{22}$$

By induction hypothesis there exists a constant $A(r-1)$ such that for each $n > A(r-1)/d(r) \geqslant A(r-1)/d(r-1)$ there is a polynomial $q_n \in \Pi_n$ such that $\hat{f}''(x)\, q_n''(x) \geqslant 0$, $x \in I$, and the inequalities (5)–(7) hold for $\hat{f}$ and q_n (since $\hat{f}(\alpha) = 0$, increasing the constant in (5) we can assume that $q_n(\alpha) = 0$).

Now we fix $n > \max\{A(r-1)/d(r), 50/(y_2 - \alpha), 50/(\alpha + 1)\}$ and consider corresponding decomposition of $[-1, 1]$: $I = \bigcup_{j=1}^n I_j = \bigcup_{j=1}^n [x_j, x_{j-1}]$. Let index j_0 be such that $\alpha \in [x_{j_0}, x_{j_0-1})$. Then $x_{j_0+3} \geqslant -1$ and $x_{j_0-4} \leqslant y_2$, i.e., $[-1, \alpha]$ and $[\alpha, y_2]$ contain at least three intervals I_j each. This implies, in particular, that $\varphi(\alpha) \geqslant n^{-1}$ and, therefore, $2\varphi(\alpha) \geqslant n\Delta_n(\alpha)$.

Now we consider the algebraic polynomial $p_n(x) := \int_\alpha^x p_n'(y)\, dy$ such that

$$p_n'(x) = (q_n'(x) - q_n'(\alpha))\, V_n(x) + q_n'(\alpha)\, W_n(x),$$

and show that it is possible to choose polynomials $V_n(x)$ and $W_n(x)$ so that p_n is coconvex with f and the inequalities (5)–(7) are satisfied. We claim that the following properties of V_n and W_n are sufficient for coconvexity of p_n with f:

(i) $V_n(x)\,\mathrm{sgn}_\alpha(x) \geqslant 0$, $x \in I$,

(ii) V_n' is copositive with $(q_n'(x) - q_n'(\alpha))\, q_n''(x)\,\mathrm{sgn}_\alpha(x)$, i.e., $(q_n'(x) - q_n'(\alpha))\, q_n''(x)\, V_n'(x)\,\mathrm{sgn}_\alpha(x) \geqslant 0$, $x \in I$,

(iii) W_n' is copositive with $f''(x)\,\mathrm{sgn}(q_n'(\alpha))$, i.e., $f''(x)\, W_n'(x)\,\mathrm{sgn}(q_n'(\alpha)) \geqslant 0$, $x \in I$.

Indeed, using these properties, the inequality $\hat{f}''(x)\, q_n''(x) \geqslant 0$ and the definition of $\hat{f}$ we have

$$\begin{aligned} &\mathrm{sgn}\{p_n''(x)\, f''(x)\} \\ &\quad = \mathrm{sgn}\{(q_n'(x) - q_n'(\alpha))\, V_n'(x)\, f''(x) \\ &\qquad + q_n''(x)\, V_n(x)\, f''(x) + q_n'(\alpha)\, W_n'(x)\, f''(x)\} \end{aligned}$$

$$\geqslant \operatorname{sgn}\{(q_n'(x)-q_n'(\alpha))\,V_n'(x)\,f''(x)+q_n''(x)\,V_n(x)\,f''(x)\}$$
$$=\operatorname{sgn}\{(q_n'(x)-q_n'(\alpha))\,V_n'(x)\,q_n''(x)\,\operatorname{sgn}_\alpha(x)$$
$$+(q_n''(x))^2\,V_n(x)\,\operatorname{sgn}_\alpha(x)\}\geqslant 0.$$

Therefore, it is sufficient to construct polynomials $V_n(x)$ and $W_n(x)$ which satisfy conditions (i)–(iii) and also (as we will see later) sufficiently approximate $\operatorname{sgn}_\alpha(x)$.

Using Lemma 6 we conclude that the polynomial

$$W_n(x):=\begin{cases} Q_{j_0+2}(n,\mu,x_{j_0+2};\varnothing;y_1,...,y_r)(x) & \text{if } q_n'(\alpha)\geqslant 0,\\ Q_{j_0-2}(n,\mu,x_{j_0-2};y_1;y_2,...,y_r)(x) & \text{if } q_n'(\alpha)<0.\end{cases}$$

satisfies condition (iii).

Indeed, it is clear that $f''(x)$ is copositive with $\prod_{i=1}^r(y_i-x)$. Lemma 6 yields that if $q_n'(\alpha)\geqslant 0$, then $W_n'(x)=Q_{j_0+2}'(n,\mu,x_{j_0+2};\varnothing;y_1,...,y_r)(x)$ is also copositive with $\prod_{i=1}^r(y_i-x)$ and, therefore, with $f''(x)$. If $q_n'(\alpha)<0$, then $W_n'(x)=Q_{j_0-2}'(n,\mu,x_{j_0-2};y_1;y_2,...,y_r)(x)$ is copositive with $(x-y_1)\prod_{i=2}^r(y_i-x)$ and, hence,

$$\operatorname{sgn}\{W_n'(x)\,f''(x)\}\,\operatorname{sgn}\{q_n'(\alpha)\}$$
$$=-\operatorname{sgn}\{W_n'(x)\,f''(x)\}=-\operatorname{sgn}\left\{-\prod_{i=1}^r(y_i-x)^2\right\}\geqslant 0.$$

Thus, (iii) is satisfied.

To construct $V_n(x)$, first, we note that since $q_n''(x)$ changes sign only at $y_2,...,y_r$, the function $q_n'(x)-q_n'(\alpha)$ is monotone on each of the intervals $[-1,y_2]$, $[y_r,1]$ and $[y_i,y_{i+1}]$, $2\leqslant i\leqslant r-1$. Thus, $q_n'(x)-q_n'(\alpha)$ has at most one zero in each of these intervals. Moreover, it changes sign at every zero different from y_i, $2\leqslant i\leqslant r$ (Note that $q_n'(x)-q_n'(\alpha)$ vanishes on some subinterval only if $q_n(x)$ is a linear function. Since this case is trivial, everywhere below we assume that $q_n(x)$ is a polynomial of degree $\geqslant 2$). Using this and also the inequality $q_n''(x)\leqslant 0$, $-1\leqslant x\leqslant y_2$ we conclude that the function $(q_n'(x)-q_n'(\alpha))\,q_n''(x)$ is nonpositive for $-1\leqslant x\leqslant\alpha$, nonnegative for $\alpha\leqslant x\leqslant y_2$ and has at most $2(r-1)$ changes of sign on $[y_2,1]$ (we denote these points in increasing order by $\beta_1,\beta_2,...,\beta_l$, $l\leqslant 2r-2$ and note that $\beta_1=y_2$). Hence, $(q_n'(x)-q_n'(\alpha))\,q_n''(x)$ is copositive with $(x-\alpha)\prod_{i=1}^l(\beta_i-x)$.

Now we define

$$V_n(x):=Q_{j_0}(n,\mu,\alpha;\varnothing;\beta_1,...,\beta_l)(x).$$

Condition (i) immediately follows from (20). Using (17) we conclude that $V_n'(x)$ is copositive with $\prod_{i=1}^{l} (\beta_i - x)$. Therefore,

$$\operatorname{sgn}\{(q_n'(x) - q_n'(\alpha))\, q_n''(x)\, V_n'(x)\, \operatorname{sgn}_\alpha(x)\}$$
$$= \operatorname{sgn}\left\{(x-\alpha)\operatorname{sgn}_\alpha(x) \prod_{i=1}^{l} (\beta_i - x)^2\right\} \geqslant 0,$$

and (ii) is also satisfied.

Thus, $p_n(x)$ is coconvex with $f(x)$ and it remains to verify the inequalities (5)–(7).

Using [5], properties of ω_φ, inequality (22) and recalling that $2\varphi(\alpha) \geqslant n\Delta_n(\alpha)$ we have the following estimates for any $x \in I$:

$$\begin{aligned}
|\hat{f}''(x)| &= |\hat{f}''(x) - \hat{f}''(\alpha)| \leqslant \omega_\varphi\left(\hat{f}'', \frac{2\,|x-\alpha|}{\varphi(\alpha)}\right) \\
&\leqslant \omega_\varphi\left(\hat{f}'', 4\,\frac{|x-\alpha| + h_{j_0}}{n\Delta_n(\alpha)}\right) \leqslant C\,\frac{|x - x_{j_0}| + h_{j_0}}{h_{j_0}}\,\omega_\varphi(\hat{f}'', n^{-1}) \\
&= C\psi_{j_0}^{-1}\omega_\varphi(\hat{f}'', n^{-1}) \leqslant C\psi_{j_0}^{-1}\omega_\varphi(f'', n^{-1})
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
|\hat{f}'(x)| &= |\hat{f}'(x) - \hat{f}'(\alpha)| = |x-\alpha|\;|\hat{f}''(\zeta)| \\
&\leqslant C\,|x-\alpha|\,\frac{|\zeta - x_{j_0}| + h_{j_0}}{h_{j_0}}\,\omega_\varphi(f'', n^{-1}) \\
&\leqslant C(|x - x_{j_0}| + h_{j_0})\,\frac{|x - x_{j_0}| + h_{j_0}}{h_{j_0}}\,\omega_\varphi(f'', n^{-1}) \\
&\leqslant Cn^{-1}\psi_{j_0}^{-2}\omega_\varphi(f'', n^{-1}),
\end{aligned} \tag{24}$$

since $|x-\alpha| \geqslant |\zeta - \alpha|$.

Now we choose μ so that all the conditions above are satisfied. For example, $\mu = 15r$ will do. However, because this choice of μ is not important we will continue to write μ keeping in mind that $\mu = \mu(r)$.

Using (23), (24) and also the inequalities $|V_n'(x)| \leqslant C(\mu)\,\psi_{j_0}^{\mu} h_{j_0}^{-1}$, $|W_n'(x)| \leqslant C(\mu)\,\psi_{j_0}^{\mu} h_{j_0}^{-1}$, $|\operatorname{sgn}_\alpha(x) - V_n(x)| \leqslant C(\mu)\,\psi_{j_0}^{\mu}$ and $|\operatorname{sgn}_\alpha(x) - W_n(x)| \leqslant C(\mu)\,\psi_{j_0}^{\mu}$, which follow from the definitions of V_n and W_n and Lemma 6 (since $\psi_{j_0} \sim \psi_{j_0 \pm i}$, $i = 1, 2$ and $|\operatorname{sgn}_\alpha(x) - \operatorname{sgn}_{x_{j_0 \pm 2}}(x)| \leqslant C(\mu)\,\psi_{j_0}^{\mu}$), we have the following estimates:

$$\begin{aligned}
&|f''(x)-p_n''(x)| \\
&\quad = |(\hat{f}''(x)-q_n''(x))\,\mathrm{sgn}_\alpha(x)+q_n''(x)(\mathrm{sgn}_\alpha(x)-V_n(x)) \\
&\qquad -(q_n'(x)-q_n'(\alpha))\,V_n'(x)-q_n'(\alpha)\,W_n'(x)| \\
&\quad \leqslant |\hat{f}''(x)-q_n''(x)|\,(1+|\mathrm{sgn}_\alpha(x)-V_n(x)|) \\
&\qquad +|\hat{f}''(x)|\,|\mathrm{sgn}_\alpha(x)-V_n(x)|+|\hat{f}'(x)-q_n'(x)|\,|V_n'(x)| \\
&\qquad +|\hat{f}'(\alpha)-q_n'(\alpha)|\,(|V_n'(x)|+|W_n'(x)|)+|\hat{f}'(x)|\,|V_n'(x)| \\
&\quad \leqslant C(r)\,\omega_\varphi(f'',n^{-1})\left\{\frac{1+\psi_{j_0}^\mu}{\min(\sqrt{y_2+1},\sqrt{1-y_r})}+\psi_{j_0}^{\mu-1}+\frac{\psi_{j_0}^\mu}{nh_{j_0}}+\frac{\psi_{j_0}^{\mu-2}}{nh_{j_0}}\right\} \\
&\quad \leqslant C(r)\,\omega_\varphi(f'',n^{-1})\left\{\frac{1}{\min(\sqrt{y_2+1},\sqrt{1-y_r})}+\frac{1}{\sqrt{1-\alpha^2}}\right\},
\end{aligned}$$

since $nh_{j_0}\geqslant\sqrt{1-\alpha^2}$. Therefore,

$$|f''(x)-p_n''(x)|\leqslant\frac{C(r)}{\min(\sqrt{\alpha+1},\sqrt{1-y_r})}\,\omega_\varphi(f'',n^{-1})=\frac{C(r)}{\sqrt{d_0}}\,\omega_\varphi(f'',n^{-1}).$$

Similarly,

$$\begin{aligned}
&|f'(x)-p_n'(x)| \\
&\quad = |(\hat{f}'(x)-q_n'(x))\,\mathrm{sgn}_\alpha(x)+q_n'(x)(\mathrm{sgn}_\alpha(x)-V_n(x)) \\
&\qquad +q_n'(\alpha))(V_n(x)-W_n(x))| \\
&\quad \leqslant C(r)\,n^{-1}\omega_\varphi(f'',n^{-1})\{1+\psi_{j_0}^\mu+\psi_{j_0}^{\mu-2}\} \\
&\quad \leqslant C(r)\,n^{-1}\omega_\varphi(f'',n^{-1})
\end{aligned}$$

and, using the identity $\int_\alpha^x q_n'(y)\,\mathrm{sgn}_\alpha(y)\,dy=q_n(x)\,\mathrm{sgn}_\alpha(x)$,

$$\begin{aligned}
&|f(x)-p_n(x)| \\
&\quad = \left|(\hat{f}(x)-q_n(x))\,\mathrm{sgn}_\alpha(x)+\int_\alpha^x q_n'(y)(\mathrm{sgn}_\alpha(y)-V_n(y))\,dy\right. \\
&\qquad \left.+q_n'(\alpha)\int_\alpha^x (V_n(y)-W_n(y))\,dy\right| \\
&\quad \leqslant C(r)\,n^{-2}\omega_\varphi(f'',n^{-1}) \\
&\qquad \times\left\{1+n\left|\int_\alpha^x (1+\psi_{j_0}(y)^{-2})\,\psi_{j_0}(y)^\mu\,dy\right|+n\left|\int_\alpha^x \psi_{j_0}(y)^\mu\,dy\right|\right\}
\end{aligned}$$

$$\leqslant C(r)\, n^{-2}\omega_\varphi(f'', n^{-1}) \left\{ 1 + n \left| \int_\alpha^x \left(\frac{h_{j_0}}{|y - x_{j_0}| + h_{j_0}} \right)^{\mu - 2} dy \right| \right\}$$

$$\leqslant C(r)\, n^{-2}\omega_\varphi(f'', n^{-1}) \left\{ 1 + nh_{j_0}^{\mu-2} \left| \int_\alpha^x (|y - \alpha| + h_{j_0})^{2-\mu}\, dy \right| \right\}$$

$$\leqslant C(r)\, n^{-2}\omega_\varphi(f'', n^{-1}) \{ 1 + nh_{j_0} \} \leqslant C(r)\, n^{-2}\omega_\varphi(f'', n^{-1}).$$

Finally, to complete the proof of Theorem 1 it is sufficient to recall that $p_n \in \Pi_{C(r)\,n}$ and use properties of ω_φ modulus.

Remark. Although, all the proofs were given in the case when f has finitely many inflection points, the considerations will not change if we allow f to be linear on some subintervals. For example, if $f''(x) = 0$ for $x \in [\alpha, \beta] \subset (-1, 1)$, it is sufficient to fix any $x_0 \in [\alpha, \beta]$ as an "inflection" point. Thus, Theorem 1 is valid for any function with finite number of convexity changes.

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